Advanced Calculus Midterm Exam

November 18, 2006

1. (12 points) Let $A = \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{Q}\}$.
   (a) Find A', the set of all cluster point of A. Justify your answer.

   Solution: Since for any $P = (x,y) \in \mathbb{R}^2$ and for any $r > 0$, the open ball neighborhood $B_P(r)$ of P, satisfies that $B_P(r) \cap A \setminus \{P\} \neq \emptyset$, $A' = \mathbb{R}^2$.

   (b) Find the boundary $\mathrm{Bd}(A)$. Justify your answer.

   Solution: Since for any $P = (x,y) \in \mathbb{R}^2$ and for any $r > 0$, the open ball neighborhood $B_P(r)$ of P, satisfies that $B_P(r) \cap A \neq \emptyset$, and $B_P(r) \cap \left(\mathbb{R}^2 \setminus A\right) \neq \emptyset$, $\mathrm{Bd}(A) = \mathbb{R}^2$.

   (c) Find the interior $\mathrm{Int}(A)$. Justify your answer.

   Solution: Since for any $P = (x,y) \in \mathbb{R}^2$ and for any $r > 0$, the open ball neighborhood $B_P(r)$ of P, satisfies that $B_P(r) \cap \left(\mathbb{R}^2 \setminus A\right) \neq \emptyset$, $\mathrm{Int}(A) = \emptyset$.

2. (10 points) Let $\{U_a\}_{a \in \mathscr{A}}$ be a collection of open subsets in $\mathbb{R}^p$. Show that the set $U = \cup_{a \in \mathscr{A}} U_a$ is open in $\mathbb{R}^p$.

   Solution: For any $q \in U$, $q \in U_a$ for some $a \in \mathscr{A}$. Since U_a is open, there exists a $r > 0$, such that $q \in B_q(r) \subset U_a \subset U$ which implies that U is open in $\mathbb{R}^p$.

3. (10 points) Show that the set $\Delta = \{(x,x) | x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$. Note that the set Δ represents the graph of the function $f(x) = x$ in $\mathbb{R}^2$.

   Solution: For any $P = (a,b) \notin \Delta$, letting $r = \dfrac{|b-a|}{2}$, we have $r > 0$.
   <u>**Claim:**</u> $\Delta \cap B_P(r) = \emptyset$
   The claim obviously implies that $\mathbb{R}^2 \setminus \Delta$ is open in $\mathbb{R}^2$. Hence, Δ is closed in $\mathbb{R}^2$.
   <u>**Proof of Claim:**</u> Observe that
   $B_P(r) \subset Q = \{(x,y) \mid 0 \leq |x-a| < r;\, 0 \leq |x-b| < r\}$, i.e. Q is a square of length $2r$with center P.
   Suppose $\Delta \cap Q \neq \emptyset$ and (c,c) is a point in $\Delta \cap Q$, then we have
   $0 \leq |c-a| < r$ and $0 \leq |c-b| < r$.
   But, by using the triangle inequality, we have
   $|b-a| = |c-a-(c-b)| \leq |c-a| + |c-b| < r + r = 2r = |b-a|$ which is contradiction.
   Therefore, $\Delta \cap Q = \emptyset$. Since $B_P(r) \subset Q$, we have $\Delta \cap B_P(r) = \emptyset$.

4. (a) (10 points) Let $A \subset \mathbb{R}^p$. Show that $\overline{A} = A \cup A'$, where A' is the set of all cluster point of A.

   Solution: For any $x \notin A \cup A'$, $x \notin A$, and $x \notin A'$. From the definition of $x \notin A'$, there is an open neighborhood U of x, such that $U \cap A = U \cap A \setminus \{x\} = \emptyset$ which also implies that $U \cap A' = \emptyset$. Therefore, U is an open neighborhood of x satisfying $U \cap \left(A \cup A'\right) = \emptyset$, i.e. $x \in U \subset \left(A \cup A'\right)^c$ and $\left(A \cup A'\right)^c$ is open or $A \cup A'$ is a closed. Since $A \subset A \cup A'$, $A \cup A'$ is a closed set containing A. The definition of $\overline{A}$ implies that $\overline{A} \subseteq A \cup A'$.
   On the other hand, for any $x \notin \overline{A}$, $x \notin A$ and, since $\overline{A}$ is closed, there is an open neighborhood U of x such that $U \subseteq \left(\mathbb{R}^p \setminus \overline{A}\right)$ which implies that $U \cap A = \emptyset$, i.e. $x \notin A'$. This implies that $A \cup A' \subseteq \overline{A}$.
   Hence, we have $\overline{A} = A \cup A'$.

(b) (4 points) Let $F \subset \mathbb{R}^p$ be a closed subset. By using (a), prove that F is closed if and only if it contains all its cluster points.

Solution: If F is closed, since $F \subseteq F$, we have $\overline{F} = F$. By using part (a), we get $F = \overline{F} = F \cup F'$ which implies that $F' \subseteq F$.
Conversely, if $F' \subseteq F$, by part (a), we have $F = F \cup F' = \overline{F}$. Since $\overline{F}$ is closed, F is closed.

5. (10 points) Let $A \subset \mathbb{R}^p$. Show that $\mathrm{Bd}(A) = \overline{A} \cap \overline{A^c}$, where $A^c = \mathbb{R}^p \setminus A$ denotes the complementary set of A in $\mathbb{R}^p$.

Solution: By $4(a)$, $\overline{A} = A \cup A'$, and $\overline{A^c} = A^c \cup (A^c)'$, we have
$\overline{A} \cap \overline{A^c}$
$= \left(A \cup A'\right) \cap \left(A^c \cup (A^c)'\right) = \left(A \cap A^c\right) \cup \left(A' \cap A^c\right) \cup \left(A \cap (A^c)'\right) \cup \left(A' \cap (A^c)'\right)$
$= \left(A' \cap A^c\right) \cup \left(A \cap (A^c)'\right) \cup \left(A' \cap (A^c)'\right)$.
Hence, a point $x \in \overline{A} \cap \overline{A^c}$
if and only if $x \in \left(A' \cap A^c\right)$ or $x \in \left(A \cap (A^c)'\right)$ or $x \in \left(A' \cap (A^c)'\right)$
if and only if for any open neighborhood U of x $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$
if and only if $x \in \mathrm{Bd}(A)$.
Therefore, we have $\overline{A} \cap \overline{A^c} = \mathrm{Bd}(A)$.

6. (10 points) Let X and Y be non-empty sets and let $f : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Prove that $\sup_y \inf_x f(x,y) \leq \inf_x \sup_y f(x,y)$.

Solution: For each $x \in X$, and $y \in Y$, since $\inf_x f(x,y) \leq f(x,y)$, we have $\sup_y \inf_x f(x,y) \leq \sup_y f(x,y)$, for all $x \in X$ which implies that $\sup_y \inf_x f(x,y) \leq \inf_x \sup_y f(x,y)$.

7. (10 points) Let $K \subset \mathbb{R}^p$ be a compact subset. Prove directly (without using Heine-Borel theorem) that K is closed and bounded.

Solution: <u>**Proof of closedness:**</u> For any $x \notin K$, the set $\{\mathbb{R}^p \setminus \overline{B_x(\frac{1}{n})} \mid n \in \mathbb{N}\}$ is an open covering of K. The compactness of K implies that there is a $m \in \mathbb{N}$ such that $K \subseteq \cup_{n=1}^m \mathbb{R}^p \setminus \overline{B_x(\frac{1}{n})} = \mathbb{R}^p \setminus \overline{B_x(\frac{1}{m})}$ which implies that $B_x(\frac{1}{m}) \cap K = \emptyset$ and K^c is open or K is closed.
<u>**Proof of boundedness:**</u> Let o denote the origin of $\mathbb{R}^p$. The set $\{B_o(n) \mid n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}^p$, hence it is an open covering of K. The compactness of K implies that there is a $m \in \mathbb{N}$ such that $K \subset \cup_{n=1}^m B_o(n) = B_o(m)$ which implies that K is bounded.

8. (10 points) Let C be an open, connected subset of $\mathbb{R}^p$, and W be any set satisfying that $C \subseteq W \subseteq \overline{C}$. Show that W is connected.

Solution: Suppose that W is disconnected and A, B are two open sets separating W. Since $A \cap W \neq \emptyset$ and $W \subseteq \overline{C} = C \cup C'$, we have $\emptyset \neq A \cap \overline{C} = \left(A \cap C\right) \cup \left(A \cap C'\right)$ which implies that $A \cap C \neq \emptyset$. By using a similar argument, we have $B \cap C \neq \emptyset$. This implies that A, B separate C which contradicts to that C being connected. Hence, W is connected.

9. Let C_1, C_2 be open, connected subsets of $\mathbb{R}^p$. Assume that $C_1 \cap C_2 \neq \emptyset$.

   (a) (4 points) Is it true that $C_1 \cap C_2$ is always connected? Why?

   Solution: No, consider $C_1 = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4,\ y > -\frac{1}{2}\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4,\ y < \frac{1}{2}\}$, then both C_1, C_2 are open, connected subsets of $\mathbb{R}^2$, while $C_1 \cap C_2$ is disconnected.

   (b) (10 points) Show that $C_1 \cup C_2$ is connected.

   Solution: Suppose that $C_1 \cup C_2$ is disconnected and A, B are two open sets separating $C_1 \cup C_2$. Let x be a point in $C_1 \cap C_2$, and assume that $x \in A$. Then $A \cap C_1 \neq \emptyset$ and $A \cap C_2 \neq \emptyset$. Since $C_1 \cup C_2 = A \cup B$, we get either $C_1 \cap B \neq \emptyset$ or $C_2 \cap B \neq \emptyset$ which would imply A, B separate either C_1 or C_2, respectively. Either case contradicts to that both C_1, C_2 are connected. Hence, $C_1 \cup C_2$ is connected.